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Biofluiddynamics of balistiform and gymnotiform locomotion. Part 2. The pressure distribution arising in two-dimensional irrotational flow from a general symmetrical motion of a flexible flat plate normal to itself

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When, for an otherwise unbounded fluid, the unique irrotational flow compatible with the instantaneous motion of an immersed body has been calculated, it is straightforward to deduce the pressure field from the unsteady form of Bernoulli's equation if the body is rigid. On the other hand, if the body is flexible, a somewhat subtle analysis is required to determine the time derivative of velocity potential, $\partial\phi/\partial t$, which occurs in that equation. This is because no simple relationship exists between the instantaneous form of ϕ and its form at a nearby instant.

In the case of two-dimensional flow, however, the two forms of ϕ for a flexible body may be related, not in general by a simple translational and/or rotational mapping as for a rigid-body motion, but by a conformal mapping. The example of a flexible flat plate is used here to illustrate this approach to calculating the pressure field.

In the analysis of balistiform motion by elongated-body theory (Lighthill & Blake 1990), one part of the propulsive force on the fish has magnitude equal to P, the area integral of the pressure field just described. This area integral is shown in §3 below to take a simple form $\bar{U}M-E$ in terms of the flow's momentum M and kinetic energy E per unit length and a certain weighted average $\bar{U}$ of the plate's velocity normal to itself. Although, in the case of motile fins attached to a rigid body of much greater depth, M was found (Lighthill & Blake 1990) to take an enhanced value, no such enhancement is found either for the product $\bar{U}M$ or for E, so that P itself is also not enhanced. For the relevance of these findings to the efficiency of balistiform motion, see Lighthill & Blake (1990).

1. Introduction

For the biological background to this analysis, see Lighthill & Blake (1990, hereafter referred to as Part 1). The present paper draws attention first of all to certain characteristic difficulties of deducing the pressure fields that arise in irrotational flows generated by movements of a *flexible* body. It continues with a two-dimensional irrotational-flow calculation of the pressure distribution due to a general symmetrical flexing motion of a flat plate, and proceeds to derive a quantity needed in elongated-body theories of balistiform locomotion (see Part 1); namely, the area integral of the pressure over the whole flow field, with the semi-convergent integral made precise (for reasons familiar from added-mass theory) by the integration in the direction normal to the plate preceding that in the direction parallel to the plate.

The classical formula for the excess pressure p_e (excess over hydrostatic pressure) in an unsteady irrotational flow that vanishes in the far field is

$$p_e = -\rho\,\partial\phi/\partial t - \tfrac{1}{2}\rho(\nabla\phi)^2, \tag{1}$$

where ρ is the density and ϕ the velocity potential. In (1), the presence of the $\partial\phi/\partial t$ term means that the pressure depends not merely on the velocity potential associated with the instantaneous motion of the boundary but also on how the velocity potential changes in a short time interval during which the boundary's position is changed.

This gives rise to no difficulties if the boundary remains rigid. For example, if it undergoes a uniform translation in the y-direction with velocity U, then the whole flow field is similarly translated so that the potential ϕ takes the form

$$\phi = \phi(x, y-Ut), \tag{2}$$

giving

$$\partial\phi/\partial t = -U\,\partial\phi/\partial y. \tag{3}$$

In this case, the classical formula (1) becomes

$$p_e = +\rho U\,\partial\phi/\partial y - \tfrac{1}{2}\rho(\nabla\phi)^2. \tag{4}$$

Its area integral in the sense noted above can be written as U times the y-component of fluid momentum M minus the kinetic energy E; giving

$$UM - E = U(mU) - \tfrac{1}{2}mU^2 = +\tfrac{1}{2}mU^2. \tag{5}$$

Here, the added mass m is defined so that the fluid kinetic energy is $\tfrac{1}{2}mU^2$ while the 'impulse' of the irrotational flow field in the y-direction is mU; which, in addition, can be written as the integral of $\rho\,\partial\phi/\partial y$ over the fluid region (that is, as the y-component of momentum) provided that the integration with respect to y is carried out before the integration with respect to x. In physical terms this procedure yields the y-component of momentum between two parallel planes $x = \pm X$ where X is large; an important quantity because its rate of change is simply the y-component of the force with which the body acts on the fluid, being uncontaminated by the overall pressure force acting across those planes (which cannot have any y-component).

When the shape of the body is being altered by flexure, no simple rule such as the 'translation' formula (2) is available for relating the flow field at time t to that at $t = 0$. On the other hand, for two-dimensional irrotational flows we have the option of replacing translation by an appropriate conformal mapping. This idea is used below to obtain an expression for the pressure field associated with a general symmetrical motion of a flexible flat plate, and to calculate its area integral in the above sense. Choice of this approach to the problem through a conformal mapping was made after it was recognized that standard methods which can sometimes be used for solving a boundary-value problem on a slightly distorted boundary (methods utilizing the derivative of the velocity on the original position of the boundary) are of uncertain validity where the velocity field is singular as at the edges of a flat plate.

2. Preliminary calculation for a circular cylinder

A considerably simpler flow, produced by flexing of the boundary of a circular cylinder, is used to introduce the conformal-mapping idea. If $Z = X + \mathrm{i}Y$ is a complex variable in the fluid region outside a circular cylinder $|Z| = 1$, then the conformal mapping

$$Z_t = Z + i\epsilon(t)\, Z^{-2n} \tag{6}$$

(where n is a non-negative integer and $\epsilon(t)$ is small and real) maps that region into a region in the Z_t plane subject to a slight distortion symmetrical about the Y-axis and antisymmetrical about the X-axis. Its boundary is given parametrically by

$$Z = e^{i\theta}, \quad Z_t = e^{i\theta}[1 + i\epsilon(t)\, e^{-(2n+1)i\theta}] \tag{7}$$

so that to first order in ϵ the radial distortion of the boundary is

$$|Z_t| - 1 = \epsilon(t) \sin(2n+1)\,\theta; \tag{8}$$

thus, the case $n = 0$ corresponds to a uniform translation with displacement $\epsilon(t)$, while positive values of n correspond to various modes of deformation.

If the complex potential

$$w = \phi + i\psi \tag{9}$$

(where ψ is the stream function) is derived in the distorted fluid-flow region by applying the mapping (6) to a complex potential w determined in the undisturbed region $|Z| = 1$, then we need to be able to calculate

$$(\partial w/\partial t)_{Z_t}. \tag{10}$$

This expression utilizes the conventional bracket notation which signifies rate of change of w with time keeping the position Z_t constant. Evidently, the derivative $\partial\phi/\partial t$ occurring in expression (1) for excess pressure is the real part of this derivative (10) in which position is held constant.

Now, the classical formula for partial differentiation after change of variables between (Z, t) and (Z_t, t) gives

$$(\partial w/\partial t)_Z = (\partial w/\partial t)_{Z_t} + (\partial w/\partial Z_t)_t\,(\partial Z_t/\partial t)_Z. \tag{11}$$

In cases when the rate of deformation is not itself varying with time the left-hand side of (11) may be zero, while in other cases its integral may be zero (see below). Then the important term contributing to the complex expression (10) (whose real part is $\partial\phi/\partial t$) is

$$-(\partial Z_t/\partial t)_Z\,(\partial w/\partial Z_t)_t = -i\dot{\epsilon} Z^{-2n}(\partial w/\partial Z_t)_t \tag{12}$$

which in the undisturbed state (at $t = 0$) takes the value

$$-i\dot{\epsilon} Z^{-2n}\, \partial w/\partial Z, \tag{13}$$

corresponding directly to the right-hand side of (3).

Here, the complex potential (9) at time $t = 0$ associated with the motion (8) of the boundary satisfies

$$\partial\phi/\partial r = \dot{\epsilon} \sin(2n+1)\,\theta \quad \text{on} \quad r = 1, \quad \text{where} \quad Z = r\, e^{i\theta}, \tag{14}$$

so that

$$\phi = -(2n+1)^{-1}\, \dot{\epsilon} r^{-(2n+1)} \sin(2n+1)\,\theta \tag{15}$$

and

$$w = -i(2n+1)^{-1}\, \dot{\epsilon} Z^{-2n-1}. \tag{16}$$

The associated contribution to (13) is

$$\dot{\epsilon}^2 Z^{-4n-2} \tag{17}$$

of which the real part, amounting to

$$\dot{\epsilon}^2 r^{-4n-2} \cos(4n+2)\,\theta, \tag{18}$$

is the contribution to $\partial\phi/\partial t$.

The integral of (18) over the fluid region

$$r > 1, \quad 0 < \theta < 2\pi \tag{19}$$

is absolutely convergent for $n > 0$ and takes the value zero (since the integral of $\cos(4n+2)\theta$ with respect to θ is zero). However, for $n = 0$ it is semi-convergent and in terms of X and Y can be written as the integral of

$$\frac{\dot{\epsilon}^2(X^2-Y^2)}{(X^2+Y^2)^2} = \dot{\epsilon}^2\frac{\partial}{\partial Y}\left[\frac{Y}{X^2+Y^2}\right]. \tag{20}$$

Proceeding to integrate (20) according to the rule that the integration with respect to Y must be carried out first, we obtain the result

$$\dot{\epsilon}^2\oint\frac{Y\,\mathrm{d}X}{X^2+Y^2} = -\pi\dot{\epsilon}^2, \tag{21}$$

where the loop integral is taken around $X^2+Y^2 = 1$ in the positive sense.

Needless to say, this calculation for the case $n = 0$ of uniform translation at velocity $U = \dot{\epsilon}$ has simply recovered the result (5) for the integrated pressure; thus, it gives a contribution

$$\rho\pi U^2 = mU^2 = UM \tag{22}$$

to the area integral of $-\rho\,\partial\phi/\partial t$ since the added mass for a circular cylinder of unit radius is $\rho\pi$. For positive values of n, however, the part $-\rho\,\partial\phi/\partial t$ of the excess pressure (1) has zero area integral so that in these cases the integrated pressure is minus the kinetic energy, and is readily calculated from (15) as

$$-\rho\pi\dot{\epsilon}^2(2n+1)^{-1}. \tag{23}$$

This example illustrates the possibility that the contribution to thrust from such an area integral of the pressure could be negative; although we shall find it to be positive in the case (§4) relevant to balistiform swimming.

Before proceeding to the case of the flat plate, we may briefly indicate the reason why in the above analysis it was justifiable to ignore the left-hand side of (11). In that expression the function $w(Z,t)$ represents the complex potential of the flow at time t around the distorted circular cylinder mapped onto the region outside the undistorted cylinder. Accordingly, if the rate of distortion $\dot{\epsilon}$ is unchanging at time $t = 0$, then so is $w(Z,t)$ and the left-hand side of (11) vanishes. When on the other hand $\ddot{\epsilon}$ is non-zero, the corresponding contribution to $\partial\phi/\partial t$ by (15) is

$$-(2n+1)^{-1}\,\ddot{\epsilon}r^{-(2n+1)}\sin(2n+1)\theta, \tag{24}$$

which is an odd function of $Y = r\sin\theta$ and contributes nothing to the area integral of the pressure.

3. Calculation for a flat plate

From the conclusions of §2 for a circular cylinder we move to calculations for a flat plate by applying the classical conformal mapping

$$z = Z + Z^{-1} \tag{25}$$

which maps the circle $|Z| = 1$ onto a strip

$$-2 < x < 2, \quad y = 0, \quad \text{since} \quad z = x+\mathrm{i}y = 2\cos\theta \quad \text{for} \quad Z = \mathrm{e}^{\mathrm{i}\theta}. \tag{26}$$

The variable Z_t defined by equation (6) and having the boundary values (7) has these boundary values, to first order in ϵ, mapped onto

$$z_t = Z_t + Z_t^{-1} = 2\cos\theta - 2\epsilon\, e^{-(2n+1)i\theta}\sin\theta, \tag{27}$$

of which the imaginary part (representing the normal displacement of the boundary) is

$$y_t = 2\epsilon \sin\theta \sin(2n+1)\theta. \tag{28}$$

This form (28) for the normal displacement of the flat-plate boundary (26) allows us to characterize a general symmetrical displacment of the flat plate,

$$y = \epsilon(t) f(x) \quad (-2 < x < 2), \tag{29}$$

as a linear combination of such terms (28) vanishing at $x = \pm 2$ (where $\theta = 0$ or π) with a uniform translation

$$y = \epsilon(t)\, b \quad \text{where} \quad b = f(2) = f(-2). \tag{30}$$

Specifically, it allows us to use new mappings as follows, defined in Fourier-series form.

Generalizing (6), we put

$$Z_t = Z + i\epsilon(t) \sum_0^\infty a_n Z^{-2n}, \tag{31}$$

where $a_0, a_1, \ldots$ are real coefficients, and write

$$z_t = i\epsilon(t)\, b + Z_t + Z_t^{-1}. \tag{32}$$

The value of (32) on the boundary (26), to first order in ϵ, has imaginary part

$$y_t = \epsilon(t)\left[b + 2\sin\theta \sum_0^\infty a_n \sin(2n+1)\theta\right]. \tag{33}$$

Thus, the general displacement (29) can be so described provided that the function

$$[f(x) - b]/(2\sin\theta), \tag{34}$$

which by (30) is regular at $\theta = 0$ and π, has the Fourier-series representation

$$\sum_0^\infty a_n \sin(2n+1)\theta. \tag{35}$$

The chain of three conformal mappings (32), (31) and (25) maps the fluid region around the displaced flat plate (represented by the complex variable z_t) into the fluid region around the undisplaced flat plate (represented by z). By analogy with (11) we calculate the term $\partial\phi/\partial t$ in the excess pressure (1) as

$$\partial\phi/\partial t = \mathrm{Re}\,[(\partial w/\partial t)_{z_t}], \tag{36}$$

where

$$(\partial w/\partial t)_z = (\partial w/\partial t)_{z_t} + (\partial w/\partial z_t)_t\,(\partial z_t/\partial t)_z, \tag{37}$$

and find as before that the left-hand side of this equation can be taken as zero when $\ddot{\epsilon} = 0$, and otherwise as an odd function of y making no contribution to the area integral of the pressure. At time $t = 0$, then, the important contribution to $\partial\phi/\partial t$ is the real part of

$$-(\partial z_t/\partial t)(\partial w/\partial z). \tag{38}$$

Here, the value of $\partial z_t/\partial t$ to first order in ϵ, by (31) and (32), is

$$\partial z_t/\partial t = \mathrm{i}\dot{\epsilon}\left[b+(Z-Z^{-1})\sum_0^{\infty} a_n Z^{-(2n+1)}\right] \tag{39}$$

and it is now necessary to obtain $\partial w/\partial z$.

The normal velocity on the plate (33) can be written as the time derivative of the right-hand side,

$$(\partial\phi/\partial y)_{y=0} = \dot{\epsilon}\left\{b+\sum_0^{\infty} a_n[\cos 2n\theta-\cos 2(n+1)\theta]\right\}, \tag{40}$$

so that the condition corresponding to (14) in the circle plane is

$$\begin{aligned}(\partial\phi/\partial r)_{r=1} &= (2\sin\theta)(\partial\phi/\partial y)_{y=0}\\ &= \dot{\epsilon}\left\{2b\sin\theta+\sum_0^{\infty} a_n[-\sin(2n-1)\theta+2\sin(2n+1)\theta-\sin(2n+3)\theta]\right\}.\end{aligned} \tag{41}$$

To write down the corresponding value of ϕ for general r we have to treat with special care the term for $n = 0$ where the term in square brackets becomes $3\sin\theta-\sin 3\theta$. We obtain

$$\begin{aligned}\phi = \dot{\epsilon}\Big\{&-r^{-1}(2b+3a_0)\sin\theta+\tfrac{1}{3}a_0 r^{-3}\sin 3\theta\\ &+\sum_1^{\infty} a_n[(2n-1)^{-1}r^{-(2n-1)}\sin(2n-1)\theta\\ &-2(2n+1)^{-1}r^{-(2n+1)}\sin(2n+1)\theta+(2n+3)^{-1}r^{-(2n+3)}\sin(2n+3)\theta]\Big\}\end{aligned} \tag{42}$$

and therefore

$$\begin{aligned}w = \mathrm{i}\dot{\epsilon}\Big\{&-(2b+3a_0)Z^{-1}+\tfrac{1}{3}a_0 Z^{-3}\\ &+\sum_1^{\infty} a_n[(2n-1)^{-1}Z^{-(2n-1)}-2(2n+1)^{-1}Z^{-(2n+1)}+(2n+3)^{-1}Z^{-(2n+3)}]\Big\}.\end{aligned} \tag{43}$$

Finally, we infer from (39) and (43) that the product (38) whose real part represents the essential contribution to $\partial\phi/\partial t$ takes the form

$$\begin{aligned}\dot{\epsilon}^2\left[b+\sum_0^{\infty} a_n(Z^{-2n}-Z^{-(2n+2)})\right]\Big\{&(2b+3a_0)Z^{-2}\\ &-a_0 Z^{-4}-\sum_1^{\infty} a_n[Z^{-2n}-2Z^{-(2n+2)}+Z^{-(2n+4)}]\Big\}.\end{aligned} \tag{44}$$

When the two series in (44) are multiplied together, they yield a Laurent series of which the first term, in Z^{-2}, has real part

$$\dot{\epsilon}^2(b+a_0)(2b+3a_0-a_1)r^{-2}\cos 2\theta \tag{45}$$

and the others are terms in $Z^{-4}, Z^{-6}, \ldots$ with real parts proportional to

$$r^{-4}\cos 4\theta, \quad r^{-6}\cos 6\theta, \ldots. \tag{46}$$

Now when we seek the area integral

$$\int(\partial\phi/\partial t)\,\mathrm{d}x\,\mathrm{d}y \tag{47}$$

over the fluid region outside the flat plate (26), we can obtain it via the mapping (25) as the corresponding area integral

$$\int (\partial\phi/\partial t)\,|\mathrm{d}z/\mathrm{d}Z|^2\,\mathrm{d}X\,\mathrm{d}Y \tag{48}$$

over the fluid region outside the circular cylinder. Here, by (25),

$$|\mathrm{d}z/\mathrm{d}Z|^2 = 1-2r^{-2}\cos 2\theta + r^{-4} \quad \text{where} \quad Z = r\,\mathrm{e}^{\mathrm{i}\theta}. \tag{49}$$

Accordingly, the term (45) makes a contribution to the integral (48), with $|\mathrm{d}z/\mathrm{d}Z|^2$ substituted from (49), equal to

$$-2\pi\dot{\epsilon}^2(b+a_0)(2b+3a_0-a_1); \tag{50}$$

where a contribution $-\pi\dot{\epsilon}^2$ comes from the calculation (21) of the semi-convergent integral of $\dot{\epsilon}^2 r^{-2}\cos 2\theta$, while an equal contribution comes from the absolutely convergent integral of

$$\int_1^\infty\int_0^{2\pi} \dot{\epsilon}^2(-2r^{-4}\cos^2 2\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta = -\pi\dot{\epsilon}^2. \tag{51}$$

By contrast, the remaining terms (46) when substituted in (48) with $|\mathrm{d}z/\mathrm{d}Z|^2$ given by (49) yield absolutely convergent integrals all of which take the value zero. This is because the integral with respect to θ of any of those terms, either by itself or multiplied by the $\cos 2\theta$ in (49), is equal to zero.

A physical interpretation can be given to the value (50) calculated for the integral (47) when we take into account the fact that the Fourier-series (35) represents the function (34). This gives

$$b+a_0 = b+2\pi^{-1}\int_0^\pi \tfrac{1}{2}[f(x)-b]\,\mathrm{d}\theta = \pi^{-1}\int_0^\pi f(x)\,\mathrm{d}\theta \tag{52}$$

and

$$a_1-a_0 = 2\pi^{-1}\int_0^\pi \tfrac{1}{2}[f(x)-b]\,[(\sin 3\theta-\sin\theta)/\sin\theta]\,\mathrm{d}\theta$$

$$= 2\pi^{-1}\int_0^\pi f(x)\cos 2\theta\,\mathrm{d}\theta, \tag{53}$$

so that

$$2b+3a_0-a_1 = 2\pi^{-1}\int_0^\pi f(x)\,(1-\cos 2\theta)\,\mathrm{d}\theta$$

$$= 4\pi^{-1}\int_0^\pi f(x)\sin^2\theta\,\mathrm{d}\theta, \tag{54}$$

which is related to the momentum

$$M = 8\rho\dot{\epsilon}\int_0^\pi f(x)\sin^2\theta\,\mathrm{d}\theta \tag{55}$$

calculated in Part 1, equation (10).

It follows that the integral of the excess pressure (1) includes a term

$$\int (-\rho\,\partial\phi/\partial t)\,\mathrm{d}x\,\mathrm{d}y = \bar{U}M, \tag{56}$$

where from (50), (52), (54) and (55) we have written

$$\bar{U} = \dot{\epsilon}(b+a_0) = \pi^{-1}\int_0^\pi \dot{\epsilon}f(x)\,\mathrm{d}\theta \tag{57}$$

as a weighted mean of the velocity of the flat plate normal to itself, the average being taken with respect to θ. To sum up, the overall area integral of the excess pressure (1) can be written in a form of attractively simple appearance analogous to expression (5); namely,

$$\bar{U}M - E, \tag{58}$$

where E is the kinetic energy (integral of $\frac{1}{2}\rho(\nabla\phi)^2$ over the fluid region), M the y-component of momentum, and $\bar{U}$ the average of the velocity of the flat plate normal to itself taken with respect to θ.

4. Case relevant to balistiform swimming

The general theory of §3 is now applied in a case relevant to balistiform swimming. This is the case where a flat rigid body of depth $2s$ is extended by equal dorsal and ventral fins, each of depth $l-s$, so that the overall cross-section has depth $2l$. The fins are symmetrically moved, each being rigidly rotated with angular velocity ω about the stationary body.

This is a case when the velocity of the flat plate normal to itself is

$$0 \quad (0 < |x| < s), \quad \omega(|x|-s) \quad (s < |x| < l). \tag{59}$$

The average $\bar{U}$ of expression (59) with respect to θ, where

$$x = l\cos\theta, \quad s = l\cos\alpha, \tag{60}$$

is

$$\bar{U} = 2\pi^{-1}\int_0^\alpha \omega l(\cos\theta - \cos\alpha)\,\mathrm{d}\theta = 2\pi^{-1}\omega l(\sin\alpha - \alpha\cos\alpha); \tag{61}$$

where, by symmetry, the averaging is simply carried out for $0 < \theta < \frac{1}{2}\pi$.

In the limiting case of small fins, with $l-s$ small compared with s, the angle α defined in (60) becomes rather small and so the velocity $\bar{U}$ averaged with respect to θ is considerably smaller than the velocity $\omega(l-s)$ of the fin tip, by an amount

$$\bar{U}[\omega(l-s)]^{-1} = 2\pi^{-1}(\sin\alpha - \alpha\cos\alpha)(1-\cos\alpha)^{-1} \sim 4\alpha/3\pi \quad \text{as} \quad \alpha \to 0. \tag{62}$$

This reduction factor affecting the average velocity $\bar{U}$ contrasts with the behaviour of the momentum M, whose value is enhanced (as emphasized in Part 1 and Lighthill 1990) relative to its value for the fins on their own.

Specifically, the value of M calculated for this case in Part 1, equation (11), can be written

$$M = 2\rho\omega l^3(\sin\alpha - \alpha\cos\alpha - \tfrac{1}{3}\sin^3\alpha). \tag{63}$$

We infer that its non-dimensional form

$$\frac{M}{\rho\omega(l-s)^3} = \frac{2(\sin\alpha - \alpha\cos\alpha - \frac{1}{3}\sin^3\alpha)}{(1-\cos\alpha)^3} \sim \frac{32}{15\alpha} \quad \text{as} \quad \alpha \to 0. \tag{64}$$

This enhancement for small α exactly counteracts the reduction for small α shown in (62), in such a way that the product tends to a finite limit:

$$\frac{\bar{U}M}{\rho\omega^2(l-s)^4} = \frac{4(\sin\alpha - \alpha\cos\alpha)(\sin\alpha - \alpha\cos\alpha - \frac{1}{3}\sin^3\alpha)}{\pi(1-\cos\alpha)^4} \to \frac{128}{45\pi} \quad \text{as} \quad \alpha \to 0. \tag{65}$$

Figure 1 (*a*) shows that the non-dimensional form of $\bar{U}M$ specified in (65) does in fact change very little indeed as s/l varies from 0 (where $\alpha = \frac{1}{2}\pi$ and the non-dimensional form (65) takes the value $8/(3\pi) = 0.849$) to 1 (where the limiting value $128/(45\pi) = 0.905$ applies).

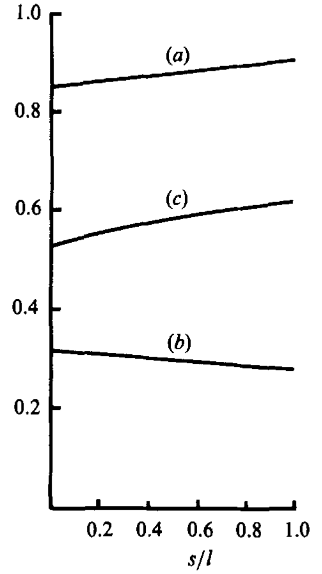

FIGURE 1. (*a*) Non-dimensional form (65) of the product of averaged velocity $\bar{U}$ and momentum M. (*b*) Non-dimensional form (66) of the kinetic energy E. (*c*) Non-dimensional form (67), obtained by subtracting (*b*) from (*a*), of the area integral of excess pressure.

Similarly, calculation of the kinetic energy E by standard methods shows that the ratio

$$E/[\rho\omega^2(l-s)^4] \tag{66}$$

changes little (figure 1*b*) as s/l varies from 0, where this ratio takes the value $1/\pi = 0.318$, to 1 where the same ratio takes a limiting value $8/(9\pi) = 0.283$. Physically, this limiting value can be simply interpreted in terms of the sum of those localized kinetic energies of the fluid motions that would be associated with each fin movement separately in the presence of an extremely deep body.

Finally, figure 1(*c*) shows the difference between the ratios (65) and (66), which by (58) gives the non-dimensional form

$$\left(\int p_e \, \mathrm{d}x \, \mathrm{d}y\right)\Big/[\rho\omega^2(l-s)^4] = (\bar{U}M-E)/[\rho\omega^2(l-s)^4] \tag{67}$$

of the area integral of excess pressure. We see that this changes little as s/l varies from 0, where the non-dimensional form (67) takes the value $5/(3\pi) = 0.531$, to $s/l = 1$ where it takes the limiting value $88/(45\pi) = 0.622$.

The conclusions of this section in the context of the analysis in Part 1 imply that in balistiform swimming the thrust developed, according to elongated-body theory, can be written in two parts, one of which (the rate of shedding, across a posterior plane, of momentum associated with fin movements) is substantially enhanced for fins of modest depth attached to a deep body, while the other term, calculated above (the area integral of excess pressure across the same plane), remains positive but relatively smaller in magnitude and is not subject to any similar enhancement. For the biological significance of these conclusions, see Part 1.

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